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A CONCISE TREATMENT OF THE SHRINK FIT WITH ELASTIC–PLASTIC HUB

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Abstract—Based on Tresca's yield criterion and the associated flow rule, the distribution of stress and displacement in a shrink fit consisting of a solid elastic shaft and an elastic–plastic hub with nonlinear isotropic hardening is derived. Since the stresses are expressed in terms of the equivalent plastic strain, the distinction between plastic and elastic regions of the hub lapses.

1. INTRODUCTION

The shrink fit consisting of a solid elastic shaft and a hardening elastic–plastic hub was treated, among others, [e.g. Kollmann (1978)] by the author (Gamer, 1987). His study is based on Tresca's yield criterion and the associated flow rule. Besides being isotropic, no restrictions are imposed on the hardening law. The stresses occurring in the plastic region of the hub contain an integral over the radius with the yield stress as a factor of the integrand. The numerical effort amounts to the calculation of the yield stress as a function of radius and the subsequent integration. However, plastic deformation in the axial direction is not admissible (Gamer and Müftü, 1990).

Following the approach of Megahed (1991), the present treatment expresses the stresses in terms of an integral over the radius with the equivalent plastic strain as a factor of the integrand. The results derived hold over the entire hub; no distinction between plastic and elastic regions need be made. In determining the distribution of stresses and displacement in a shrink fit at rest, the numerical calculation is reduced to an integration with variable upper limit for arbitrarily nonlinear hardening.

2. BASIC EQUATIONS

A material point of the hub or shaft is identified by cylindrical co-ordinates r, θ and z where the z-axis coincides with the common axis of the hub and shaft. For small strain, plane stress, $\sigma_z = 0$, independence of the circumferential direction, $\partial/\partial\theta = 0$, absence of plastic volume dilatation, $d\varepsilon^p_{ii} = 0$, and absence of plastic strain in the axial direction, $d\varepsilon^p_z = 0$, stresses, displacement and plastic strains as functions of radius may be derived in the following way (Gamer, 1991): the starting points are the equation of equilibrium in the radial direction,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{1}$$

Hooke's law,

$$E(\varepsilon_r - \varepsilon^p_r) = \sigma_r - \nu\sigma_\theta, \qquad E(\varepsilon_\theta - \varepsilon^p_\theta) = \sigma_\theta - \nu\sigma_r, \tag{2, 3}$$

and the geometric relations

$$\varepsilon_r = \frac{du}{dr}, \qquad \varepsilon_\theta = \frac{u}{r} \tag{4, 5}$$

under the stated restrictions on the plastic strains,

$$\varepsilon_\theta^p = -\varepsilon_r^p. \tag{6}$$

Substituting the stresses for the strains and the latter by the displacement and its gradient, one obtains the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1-\nu)\left(\frac{d\varepsilon_r^p}{dr} + 2\frac{\varepsilon_r^p}{r}\right), \tag{7}$$

with the solution

$$Eu = (1-\nu)Cr + (1+\nu)\frac{D}{r} + (1-\nu)Er\int\frac{\varepsilon_r^p}{r}\,dr, \tag{8}$$

where C and D are constants of integration. The displacement leads to the stresses

$$\sigma_r = C - \frac{D}{r^2} + E\int\frac{\varepsilon_r^p}{r}\,dr, \tag{9}$$

$$\sigma_\theta = C + \frac{D}{r^2} + E\int\frac{\varepsilon_r^p}{r}\,dr + E\varepsilon_r^p. \tag{10}$$

Eliminating C from the displacement (8) with the help of the radial stress (9), one arrives at

$$Eu = (1-\nu)\sigma_r r + 2\frac{D}{r} \tag{11}$$

and therefrom at

$$E\varepsilon_r^p = -E\varepsilon_\theta^p = \sigma_\theta - \sigma_r - 2\frac{D}{r^2}. \tag{12}$$

In these formulae standard notation was employed. For elastic behaviour, the three terms forming the plastic strains (12) add up to zero. It should be emphasized that the above results do not hold if both nonzero stresses have the same sign (Gamer, 1983) or if, in addition to the axial stress, the circumferential stress vanishes in a part of the plastic region (Gamer and Müftü, 1990).

3. THE SHRINK FIT

The parts of a shrink fit prior to and after assembly are shown in Fig. 1. i and z mean interference and elastic–plastic boundary radius, respectively. Equations (9)–(12) hold with different constants in the shaft (s) and the hub (h). However, since the displacement vanishes at the centre, $D_s = 0$, and

$$\sigma_r = \sigma_\theta = C_s \tag{13}$$

complies with the requirement that the plastic strains vanish throughout the shaft and that

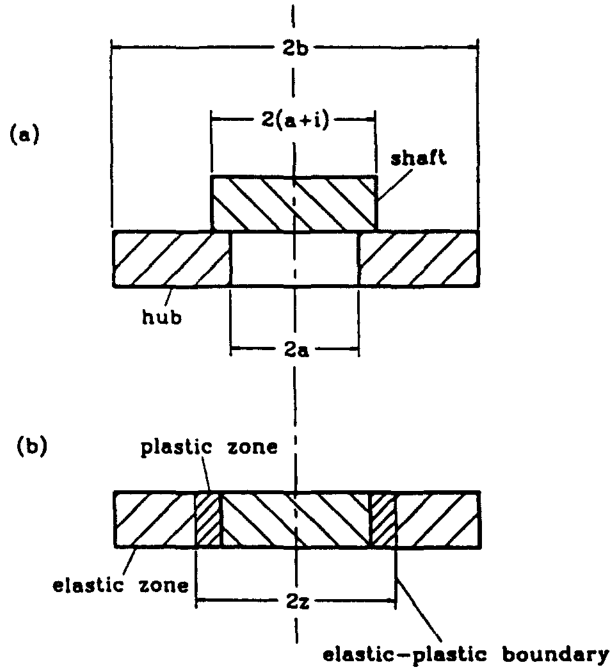

Fig. 1. Shrink fit: (a) prior to, and (b) after assembly.

the stresses are equal at its centre. Therefore, the displacement becomes

$$Eu = (1-\nu)C_s r. \tag{14}$$

In the formulation of the transition conditions at the interface, consistency with the presupposition of small strain requires that the small difference of the radius of the shaft and the inner radius of the hub prior to assembly is ignored (e.g. Biezeno und Grammel, 1953). Hence, at $r = a$, radial stress is continuous,

$$\sigma_r^s = \sigma_r^h, \tag{15}$$

and the displacement satisfies the geometrical condition

$$u^h - u^s = i. \tag{16}$$

The outer edge of the hub, $r = b$, is free of stress,

$$\sigma_r = 0. \tag{17}$$

The geometrical condition together with the continuity of radial stress yields

$$D = \tfrac{1}{2}Eai. \tag{18}$$

The subscript h is omitted.

The circumferential stress may not become compressive, $\sigma_\theta \geqslant 0$, and the radial stress does not become tensile, $\sigma_r \leqslant 0$. Hence, according to Tresca's yield condition,

$$\sigma_Y = \sigma_\theta - \sigma_r. \tag{19}$$

The yield stress σ_Y is not defined for $\sigma_\theta - \sigma_r < \sigma_0$. It is a monotonically increasing function of the equivalent plastic strain ε_{EQ},

$$\sigma_Y = \hat{\sigma}_Y(\varepsilon_{EQ}) \tag{20}$$

with $\sigma_0 = \hat{\sigma}_Y(0)$. According to the increment of plastic work,

$$\varepsilon_{EQ} = -\varepsilon_r^p = \varepsilon_\theta^p. \tag{21}$$

Considering the elastic–plastic boundary, $r = z$, one finds from (12),

$$D = \tfrac{1}{2}\sigma_0 z^2. \tag{22}$$

Comparison of (18) and (22) results in

$$z = \sqrt{\frac{E}{\sigma_0}ai}. \tag{23}$$

In other elastic–plastic small strain problems such a simple relation connecting the elastic–plastic boundary radius and load parameter does not exist. Although z can be expressed by i according to (23), it is used as a reference radius since it appeals immediately to the imagination. With (22), the equivalent plastic strain (12) becomes

$$\bar{\varepsilon}_{EQ} = -(\bar{\sigma}_\theta - \bar{\sigma}_r) + \frac{1}{\rho^2}, \tag{24}$$

where $\rho \equiv r/z$, $\bar{\varepsilon}_{EQ} \equiv E\varepsilon_{EQ}/\sigma_0$ and $\bar{\sigma}_{ij} \equiv \sigma_{ij}/\sigma_0$. In the plastic region of the hub, $\rho < 1$, $\bar{\sigma}_\theta - \bar{\sigma}_r = \bar{\sigma}_Y$ and in the elastic region of the hub, $\rho > 1$, the equivalent plastic strain vanishes. A lower limit to the plastic region and an upper limit to the elastic region are not given since the former region is fictitious in the case of the elastic hub ($z \leqslant a$) and the latter region is fictitious in the case of the fully plastic hub ($z \geqslant b$). Note that $\bar{\varepsilon}_{EQ}$ depends only on ρ.

Making use of the condition of vanishing radial stress at the free edge of the hub, one obtains from (9) and (10) the stresses

$$\bar{\sigma}_r = -\frac{1}{2}\left(\frac{1}{\rho^2} - \frac{1}{\beta^2}\right) - \int_\beta^\rho \frac{\bar{\varepsilon}_{EQ}}{\rho}\,d\rho, \tag{25}$$

$$\bar{\sigma}_\theta = \frac{1}{2}\left(\frac{1}{\rho^2} + \frac{1}{\beta^2}\right) - \int_\beta^\rho \frac{\bar{\varepsilon}_{EQ}}{\rho}\,d\rho - \bar{\varepsilon}_{EQ}. \tag{26}$$

Finally, the continuity of radial stress at the interface yields the stresses

$$\bar{\sigma}_r = \bar{\sigma}_\theta = -\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \int_\alpha^\beta \frac{\bar{\varepsilon}_{EQ}}{\rho}\,d\rho, \tag{27}$$

with $\alpha \equiv a/z$ and $\beta \equiv b/z$ in the solid shaft. The preceding results hold for the elastic hub, $z \leqslant a$, the partially plastic hub, $a < z < b$, and the fully plastic hub, $z \geqslant b$. In the second case, the lower limit of integration can be replaced by 1. In the other cases, z means the fictitious elastic–plastic boundary radius, which lies outside the hub. The integration with variable upper limit occurring in (25) and (26) need be performed once only. It applies to hubs with different degrees of plasticization as well as to hubs with different radii ratios. For the representation of results, the reference radius $z = \sqrt{Eai/\sigma_0}$ can be easily replaced by some other reference radius.

In the case where (24) cannot be solved analytically for $\bar{\varepsilon}_{EQ}$, it is not necessary to invert the function

$$\rho = [\bar{\varepsilon}_{EQ} + \bar{\bar{\sigma}}_Y(\bar{\varepsilon}_{EQ})]^{-1/2} \tag{28}$$

numerically prior to integration. Applying partial integration, one obtains

$$\int_\beta^\rho \frac{\bar{\varepsilon}_{EQ}}{\rho} d\rho = \bar{\varepsilon}_{EQ} \log \rho - \bar{\varepsilon}_{EQ}(\beta) \log \beta - \int_{\bar{\varepsilon}_{EQ}(\beta)}^{\bar{\varepsilon}_{EQ}} \log \rho \, d\bar{\varepsilon}_{EQ} \tag{29}$$

or

$$\int_1^\rho \frac{\bar{\varepsilon}_{EQ}}{\rho} d\rho = \bar{\varepsilon}_{EQ} \log \rho - \int_0^{\bar{\varepsilon}_{EQ}} \log \rho \, d\bar{\varepsilon}_{EQ} \tag{30}$$

for the partially plastic hub. The integration over the equivalent plastic strain proves advantageous if the stresses are needed at discrete radii rather than as functions of the radius.

4. EXAMPLE: PERFECTLY PLASTIC BEHAVIOUR

In the following, a partially plastic hub exhibiting perfectly plastic behaviour is considered. Equation (24) is specialized to

$$\bar{\varepsilon}_{EQ} = \left(\frac{1}{\rho^2} - 1\right) \cdot U(1-\rho), \tag{31}$$

where

$$U(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases}$$

denotes the unit step function. Hence, one obtains

$$-\int_1^\rho \frac{\bar{\varepsilon}_{EQ}}{\rho} d\rho = \left[\frac{1}{2}\left(\frac{1}{\rho^2} - 1\right) + \log \rho\right] \cdot U(1-\rho), \tag{32}$$

and the stresses in the hub become

$$\bar{\sigma}_r = -\frac{1}{2}\left(\frac{1}{\rho^2} - \frac{1}{\beta^2}\right) + \left[\frac{1}{2}\left(\frac{1}{\rho^2} - 1\right) + \log \rho\right] \cdot U(1-\rho), \tag{33}$$

$$\bar{\sigma}_\theta = \frac{1}{2}\left(\frac{1}{\rho^2} + \frac{1}{\beta^2}\right) + \left[-\frac{1}{2}\left(\frac{1}{\rho^2} - 1\right) + \log \rho\right] \cdot U(1-\rho) \tag{34}$$

or

$$\bar{\sigma}_r = -\frac{1}{2}\left(1 - \frac{1}{\beta^2}\right) + \log \rho, \tag{35}$$

$$\bar{\sigma}_\theta = \frac{1}{2}\left(1 + \frac{1}{\beta^2}\right) + \log \rho \tag{36}$$

in the plastic region, $\rho < 1$, and on the elastic–plastic boundary, $\rho = 1$, only. In the shaft, the stresses are

$$\bar{\sigma}_r = \bar{\sigma}_\theta = -\frac{1}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) + \left[\frac{1}{2}\left(\frac{1}{\alpha^2} - 1\right) + \log\alpha\right] \cdot U(1-\alpha) \tag{37}$$

or

$$\bar{\sigma}_r = \bar{\sigma}_\theta = -\frac{1}{2}\left(1 - \frac{1}{\beta^2}\right) + \log\alpha \tag{38}$$

for $\alpha \leqslant 1$. The above results also hold for the elastic hub, $z \leqslant a$, and the fully plastic hub, $z = b$. For perfectly plastic behaviour, a state of equilibrium corresponding to $z > b$ does not exist. Replacing the reference radius z by a, one recovers the well known results of Kollmann (1978).

5. EXAMPLE: NONLINEAR HARDENING OF LUDWIK-TYPE

A special case of Ludwik's hardening law, which makes the problem of the shrink fit with nonlinearly hardening hub amenable to analytical solution, is

$$\bar{\sigma}_Y = 1 + H\sqrt{\bar{\varepsilon}_{EQ}}, \tag{39}$$

where H denotes the hardening parameter. Although the dependence of $\bar{\varepsilon}_{EQ}$ on ρ exists in closed form for the hardening law under consideration, the integration over the radius will be replaced by integration over the equivalent plastic strain according to (30). Equation (28) specializes to

$$\rho = (\bar{\varepsilon}_{EQ} + H\sqrt{\bar{\varepsilon}_{EQ}} + 1)^{-1/2}. \tag{40}$$

The cases $H < 2$, $H = 2$ and $H > 2$ lead to different analytical expressions. With the abbreviations $L^2 \equiv 1 - (H/2)^2$, $P^2 \equiv (H/2)^2 - 1$ and $\varepsilon \equiv \bar{\varepsilon}_{EQ}$, one obtains for $H < 2$ or $L > 0$

$$\begin{aligned}\int_{\varepsilon(\beta)}^{\varepsilon} \log\rho \, d\varepsilon = {} & \frac{1}{2}(\varepsilon - H\sqrt{\varepsilon}) - \frac{1}{2}\left(\varepsilon + 1 - \frac{H^2}{2}\right)\log(\varepsilon + H\sqrt{\varepsilon} + 1) \\ & - HL \arcsin\frac{L}{(\varepsilon + H\sqrt{\varepsilon} + 1)^{1/2}} \\ & - \frac{1}{2}(\varepsilon(\beta) - H\sqrt{\varepsilon(\beta)}) + \frac{1}{2}\left(\varepsilon(\beta) + 1 - \frac{H^2}{2}\right)\log(\varepsilon(\beta) + H\sqrt{\varepsilon(\beta)} + 1) \\ & + HL \arcsin\frac{L}{(\varepsilon(\beta) + H\sqrt{\varepsilon(\beta)} + 1)^{1/2}},\end{aligned} \tag{41}$$

for $H = 2$

$$\begin{aligned}\int_{\varepsilon(\beta)}^{\varepsilon} \log\rho \, d\varepsilon = {} & \tfrac{1}{2}\varepsilon - \sqrt{\varepsilon} - \tfrac{1}{2}(\varepsilon - 1)\log(\varepsilon + 2\sqrt{\varepsilon} + 1) \\ & - \tfrac{1}{2}\varepsilon(\beta) + \sqrt{\varepsilon(\beta)} + \tfrac{1}{2}(\varepsilon(\beta) - 1)\log(\varepsilon(\beta) + 2\sqrt{\varepsilon(\beta)} + 1)\end{aligned} \tag{42}$$

and for $H > 2$ or $P > 0$

$$\int_{\varepsilon(\beta)}^{\varepsilon} \log \rho \, d\varepsilon = \frac{1}{2}(\varepsilon - H\sqrt{\varepsilon}) - \frac{1}{2}\left(\varepsilon + 1 - \frac{H^2}{2}\right)\log(\varepsilon + H\sqrt{\varepsilon} + 1)$$

$$+ \frac{1}{2}HP \log \frac{\left(\sqrt{\varepsilon(\beta)} + \frac{H}{2} - P\right)\sqrt{\varepsilon} + \left(\frac{H}{2} + P\right)\sqrt{\varepsilon(\beta)} + 1}{\left(\sqrt{\varepsilon(\beta)} + \frac{H}{2} + P\right)\sqrt{\varepsilon} + \left(\frac{H}{2} - P\right)\sqrt{\varepsilon(\beta)} + 1}$$

$$- \frac{1}{2}(\varepsilon(\beta) - H\sqrt{\varepsilon(\beta)}) + \frac{1}{2}\left(\varepsilon(\beta) + 1 - \frac{H^2}{2}\right)\log(\varepsilon(\beta) + H\sqrt{\varepsilon(\beta)} + 1). \quad (43)$$

Putting $\bar{\varepsilon}_{EQ}(\beta) = 0$ and making use of the inverse of (40), one recovers the stress distributions in the plastic region of the partially plastic hub reported by Gamer (1989).

6. FINAL REMARKS

The question arises whether the formalism presented above may be generalized to more complicated cases. The rotating shrink fit consisting of an elastic solid shaft and an elastic–plastic hub may be treated in the same way provided that the material parameters are equal. However, the (square of the) angular speed enters the relation connecting the equivalent plastic strain with the radius. Therefore, it is not sufficient to perform numerically the integration with variable upper limit mentioned in Section 3 once and for all and to resort to the stored results. On the contrary, for different stages of plasticization corresponding to different angular speeds, the numerical integration has to be done anew.

The case of an annular elastic shaft and/or different material parameters for the shaft and hub is more intricate since the elastic–plastic border radius cannot be determined in terms of interference and angular speed from two equations uncoupled from the other ones, but is rather determined by a complicated equation that remains after elimination of all the other unknowns and has to be solved numerically (Orçan and Gamer, 1990).

The problem of the shrink fit with a nonlinearly hardening elastic–plastic shaft is not amenable to quasi-analytical solution since the shaft suffers plastic deformation in the circumferential and axial direction in the case of a hollow shaft (Gamer, 1983) and in all three directions in the case of a solid shaft.

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